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# Variations of conjectures on counting irreducible characters of finite groups

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## 1 Introduction

Let $G$ be a finite group, $p$ a prime, $R$ the ring of algebraic integers in some finite Galois extension field $K$ of $\mathbf{Q}$ which contains enough roots of unity, $\mathcal{P}$ a prime ideal of $R$ lying over $p\mathbf{Z}$, $R_{\mathcal{P}}$ the localization of $R$ at $\mathcal{P}$, and $k$ be the residue class field $R_{\mathcal{P}}/\mathcal{P}R_{\mathcal{P}}$ of characteristic $p$.

For terminology used in modular representation theory, see [10].

Let $\mathrm{Irr}(G)$ be the set of complex irreducible characters of $G$, $e$ a primitive idempotent of the center $Z(R_{\mathcal{P}}G)$ of $R_{\mathcal{P}}G$, i.e., a block idempotent of $G$. Let $B$ be the $p$-block of $G$ corresponding to $e$.

-We say that $\chi \in \mathrm{Irr}(G)$ belongs to $B$ and write $\chi \in B$ , if $\chi(e) \neq 0$.

-We also say that an indecomposable right $kG$-module $M$ belongs to $B$, if $M\bar{e} \neq 0$, where $\bar{e}$ is the image of $e$ via the canonical epimorphism from $R_{\mathcal{P}}G$ to $kG$. We also write $M \in B$.

**Definition 1.1** *For $\chi \in \mathrm{Irr}(G)$, let $d(\chi)$ be the exponent of the highest power of $p$ in $|G|/\chi(1)$. Let $d(B) = \max\{d(\chi) |\ \chi \in B\}$.*

-For a block $B$, there exists a $p$-subgroup $D$ of $G$ such that every irreducible $kG$-module belonging to $B$ is isomorphic to a direct summand of a $kG$-module induced from a $kD$-module and that $|D| = p^{d(B)}$.

The above $D$ is unique up to $G$-conjugate and called a defect group of $B$.

-For any $\chi \in \mathrm{Irr}(G)$ and a conjugacy class $C$ of $G$, the value

$$\sum_{g\in C} \chi(g)/\chi(1) = \chi(g)|G|/\chi(1)|C_G(g)|$$

lies in $R$. The map from $Z(kG)$ to $k$ sending any $\widehat{C} = \sum_{g\in C} g$ in $Z(kG)$ to $\sum_{g\in C}\chi(g)/\chi(1) \mod \mathcal{P}$ gives a $k$-algebra homomorphism. It does not depend on the choice of $\chi \in B$ and is denoted by $\omega_B$.

**Definition 1.2** *Let $B$ be a block of $G$ and $H$ a subgroup of $G$. If a block $b$ of $H$ satisfies $\omega_B(\widehat{C}) = \omega_b(\widehat{C \cap H})$ for all conjugacy class $C$ of $G$, then we write $b^G = B$. If this is the case, we call $b^G$ the induced block and that the induced block can be*

**Theorem 1.3** (The first main theorem of Brauer) *Let $D$ be a $p$-subgroup of $G$. Then, there exists a bijection between the set of blocks of $G$ with defect group $D$ and the set of blocks of $N_G(D)$ with defect group $D$. Moreover, for a block $B$ of $G$ with defect group $D$, the corresponding block $b$ is the unique block of $N_G(D)$ with defect group $D$ such that $b^G = B$.*

# 2 Conjectures in the 20th century

For $H \leq G$, a block $B$ of $G$ and $d \in \mathbb{Z}$, let $\mathrm{Irr}(H, B, d)$ be the set of ordinary irreducible characters $\chi$ in $\mathrm{Irr}(H)$ belonging to a block $b$ of $H$ with $b^G = B$ and $d(\chi) = d$, and we denote by $k(H, B, d)$ its cardinality. Note that if a block $B$ of $G$ with defect group $D$ corresponds to a block $b$ of $N_G(D)$ via Brauer's first main theorem, then we have $\mathrm{Irr}(N_G(D), B, d) = \mathrm{Irr}(N_G(D), b, d)$ for all $d$.

**Conjecture 2.1** (Alperin-McKay Conjecture, 1970's, [9], [1]) *Suppose that a block $B$ of $G$ with defect group $D$ corresponds to a block $b$ of $N_G(D)$ via Brauer's first main theorem. Then,*

$$k(G, B, d(B)) = k(N_G(D), b, d(b)) \quad ?$$

**Remark 2.2** *For a block $B$ of $G$, Brauer's height zero conjecture asks whether $k(G, B, d) = 0$ for all $d$ with $d \neq d(B)$ if and only if $B$ has an abelian defect group.*

**Conjecture 2.3** (Broué, 1980's, [2], [3]) *Suppose that a block $B$ of $G$ with defect group $D$ corresponds to a block $b$ of $N_G(D)$ via Brauer's first main theorem and that $D$ is abelian.*

(i) (Perfect Isometry Conjecture) *Do there exist a bijection $\varphi : \mathrm{Irr}(G) \cap B \to \mathrm{Irr}(N_G(D)) \cap b$ and a map $\varepsilon : \mathrm{Irr}(G) \cap B \to \{\pm 1\}$ such that*

$$\mu(g, h) = \sum_{\chi \in \mathrm{Irr}(G) \cap B} \varepsilon(\chi)\chi(g)\varphi(\chi)(h), \quad (g \in G, h \in N_G(D))$$

*satisfies;*

*If $\mu(g, h) \neq 0$, then $g$ and $h$ are both $p$-regular or both $p$-singular.*
*Both $\mu(g, h)/|C_G(g)|$ and $\mu(g, h)/|C_{N_G(D)}(h)|$ lie in $R_{\mathcal{P}}$ ?*

(ii) (Derived Equivalence Conjecture) *Does there exist a bounded complex*

$$\mathbf{C} : \cdots \to C_{i+1} \to C_i \to \cdots$$

*of $R_{\mathcal{P}}G$-$R_{\mathcal{P}}N_G(D)$-bimodules with each $C_i$ left $R_{\mathcal{P}}G$-projective and right $R_{\mathcal{P}}N_G(D)$-projective such that $\mathbf{C} \otimes_{R_{\mathcal{P}}N_G(D)} \mathbf{C}^* \cong eR_{\mathcal{P}}G$ and $\mathbf{C}^* \otimes_{R_{\mathcal{P}}G} \mathbf{C} \cong fR_{\mathcal{P}}N_G(D)$, where $e$ and $f$ are block idempotents of $B$ and $b$, respectively ?*

(iii) (Splendid Equivalence Conjecture [13]) *Does there exist a bounded complex*

$$\mathbf{C} : \cdots \to C_{i+1} \to C_i \to \cdots$$

*of $R_{\mathcal{P}}G$-$R_{\mathcal{P}}N_G(D)$-bimodules as in* (ii) *such that each $C_i$ is a $\Delta(D)$-projective p-permutation module, where $\Delta(D) = \{(g, g^{-1}) \mid g \in D\}$ ?*

If $\varphi$ and $\varepsilon$ in (i) above exist, then we say that $\varphi$ is a *a perfect isometry* between $B$ and $b$. If $\mathbf{C}$ in (ii) above exists, then we say that $B$ and $b$ are *derived equivalent.* If $\mathbf{C}$ in (iii) above exists, then we say that $B$ and $b$ are *splendidly equivalent.*

-If the derived equivalence conjecture holds for $B$, then the perfect isometry conjecture is also true for $B$. (See §3 of [2].)

-A $\Delta(D)$-projective $p$-permutation module is by definition a direct summand of a module induced from $\Delta(D)$.

-A complex of $R_{\mathcal{P}}G$-$R_{\mathcal{P}}N_G(D)$-bimodules with the properties described in (iii) exists if and only if a complex of $kG$-$kN_G(D)$-bimodules with similar properties exists. (Rickard [13]) This fact is based on a result of Scott [18].

*A radical p-subgroup* $P$ is a $p$-subgroup of $G$ satisfying $O_p(N_G(P)) = P$, where $O_p(H)$ is the maximal normal $p$-subgroup of $H$ for a finite group $H$. *A radical p-chain*

$$\underline{C} : O_p(G) < P_1 < P_2 < \cdots < P_n$$

is a chain of $p$-subgroups $P_i$ of $G$ starting with $O_p(G)$ such that $O_p(\cap_{i=1}^{j} N_G(P_i)) = P_j$ for all $j$ with $1 \le j \le n$. Let $\mathcal{R}$ be the set of radical $p$-chains of $G$ and $\mathcal{R}/G$ a set of representatives of $G$-orbits in $\mathcal{R}$. For $\underline{C} \in \mathcal{R}$, let $N_G(\underline{C}) = \cap_{i=1}^{n} N_G(P_i)$ and $|\underline{C}| = n$. For chain normalizers, we have the following.

**Lemma 2.4** (Knörr, Robinson) *Let $\underline{C} \in \mathcal{R}$. Then, for any block $b$ of $N_G(\underline{C})$, the induced block $b^G$ can be defined.*

**Conjecture 2.5** (Dade, 1990's, [5], [6]) *Let $B$ be a block of $G$ with defect group $D$. Suppose that $D \neq \{1\}$ and $O_p(G) = \{1\}$. Then,*

$$\sum_{\underline{C} \in \mathcal{R}/G} (-1)^{|\underline{C}|} k(N_G(\underline{C}), B, d) = 0$$

*for all $d$ ?*

**Remark 2.6** *There are several forms of Dade's conjecture. They involve the number of invariant characters under the automorphism action, that of projective irreducible characters, etc.*

-Dade's conjecture (the projective form) implies the Alperin-McKay conjecture. (See Corollary 17.15 and Theorem 18.5 of [6])

-Suppose that $D$ is abelian. Then, Broué's perfect isometry conjecture implies Dade's conjecture. (See §2 of [20].)

# 3 Conjectures in the 21st century

**Definition 3.1** *For $\chi \in \mathrm{Irr}(G)$, let $r(\chi)$ be the $p'$-part of $|G|/\chi(1)$ in $(\mathbb{Z}/p\mathbb{Z})^*$ the group of units of the finite field $\mathbb{Z}/p\mathbb{Z}$.*

For $H \leq G$, a block $B$ of $G$, an integer $d$ and an element $r$ of $(\mathbb{Z}/p\mathbb{Z})^*$, let $\mathrm{Irr}(H, B, d, [\pm r])$ denote the set of irreducible characters $\chi$ in $\mathrm{Irr}(H, B, d)$ such that $r(\chi) = \pm r$, and let $k(H, B, d, [\pm r])$ denote its cardinality.

**Conjecture 3.2** (Alperin-McKay-Isaacs-Navarro Conjecture, 2001, [7]) *Let a block $B$ of $G$ with defect group $D$ corresponds to a block $b$ of $N_G(D)$ via Brauer's first main theorem. Then,*

$$k(G, B, d(B), [\pm r]) = k(N_G(D), b, d(b), [\pm r])$$

*for all $r \in (\mathbb{Z}/p\mathbb{Z})^*$ ?*

**Conjecture 3.3** (October 2001, see [17], [19]) *Let $B$ be a block of $G$ with defect group $D$. Suppose that $D \neq \{1\}$ and $O_p(G) = \{1\}$. Then,*

$$\sum_{\underline{C} \in \mathcal{R}/G} (-1)^{|\underline{C}|} k(N_G(\underline{C}), B, d, [\pm r]) = 0$$

*for all $d \in \mathbb{Z}$ and $r \in (\mathbb{Z}/p\mathbb{Z})^*$ ?*

-There are also several forms of Conjecture 3.3.

-The projective form of Conjecture 3.3 implies the Alperin-McKay-Isaacs-Navarro conjecture. The proof is similar to those of Corollary 17.15 and Theorem 18.5 of [6].

Also, the following should be noticed.

**Remark 3.4** *Suppose that $B$ is principal and $D$ is abelian. Assume further that Broué's perfect isometry conjecture holds for $B$ and the trivial character of $G$ corresponds to that of $N_G(D)$ via the perfect isometry. Then, Conjectures 3.2 and 3.3 holds for $B$.*

The reason is sketched as follows. Let $b$ be the Brauer correspondent of $B$. Suppose that there exist a bijection $\varphi : \mathrm{Irr}(G) \cap B \to \mathrm{Irr}(N_G(D)) \cap b$ and a map $\varepsilon : \mathrm{Irr}(G) \cap B \to \{\pm 1\}$ satisfying the condition. Then, we have an isomorphism

$$\tilde{\varphi} : Z(eR_{\mathcal{P}}G) \to Z(fR_{\mathcal{P}}N_G(D))$$

of $R_{\mathcal{P}}$-algebras satisfying

$$\tilde{\varphi}(e) = \sum_{\chi \in \mathrm{Irr}(G) \cap B} \frac{\varepsilon(\chi)|G|/\chi(1)}{|N_G(D)|/\varphi(\chi)(1)} e_{\varphi(\chi)},$$

where $e_{\varphi(\chi)}$ is the central idempotent of $KN_G(D)$ corresponding to $\varphi(\chi)$. (See §1 of [2].) In particular

$$\frac{\varepsilon(\chi)|G|/\chi(1)}{|N_G(D)|/\varphi(\chi)(1)}$$

is a unit in $\mathbb{Z}/p\mathbb{Z}$ independent of $\chi$. Since $\varphi(1_G) = 1_{N_G(D)}$, we must have

$$\frac{\varepsilon(\chi)|G|/\chi(1)}{|N_G(D)|/\varphi(\chi)(1)} = \pm\frac{|G|}{|N_G(D)|} \equiv \pm 1 \mod p$$

for all $\chi \in \mathrm{Irr}(G) \cap B$. Thus, $r(\chi) = \pm r(\varphi(\chi))$ for all $\chi \in \mathrm{Irr}(G) \cap B$. Hence Conjecture 3.2 follows. This observation is due to Broué. For Conjecture 3.3, we use standard pairings of chains and Brauer's third main theorem. (§2 of [20].)

Let $\mathcal{H}$ be the subgroup of $\mathrm{Gal}(K)$ defined by

$$\mathcal{H} = \{\sigma \in \mathrm{Gal}(K) \mid \mathcal{P}^\sigma = \mathcal{P}\},$$

namely, $\mathcal{H}$ is the decomposition group. For a block $B$ of $G$, let $\mathcal{H}_B$ denote the set of elements $\sigma \in \mathcal{H}$ which $\sigma$ stabilize $\mathrm{Irr}(G) \cap B$ as a set. This time, for $\sigma \in \mathcal{H}$, let $\mathrm{Irr}(H, B, d, [\pm r], \sigma)$ denote the set of $\sigma$-invariant irreducible characters in $\mathrm{Irr}(H, B, d, [\pm r])$, and $k(H, B, d, [\pm r], \sigma)$ its cardinality. Note that, if a block $B$ of $G$ with defect group $D$ corresponds to a block $b$ of $N_G(D)$ via Brauer's first main theorem, then $\mathcal{H}_B = \mathcal{H}_b$.

**Conjecture 3.5** (Alperin-McKay-Isaacs-Navarro Conjecture, July 2002, [11]) *Suppose that a block $B$ of $G$ with defect group $D$ corresponds to a block $b$ of $N_G(D)$ via Brauer's first main theorem. Then,*

$$\mathrm{Irr}(G, B, d(B), [\pm r]) \ \textit{and} \ \mathrm{Irr}(N_G(D), b, d(b), [\pm r])$$

*are isomorphic as $\mathcal{H}_B(= \mathcal{H}_b)$-sets for all $r \in (\mathbb{Z}/p\mathbb{Z})^*$ ?*

**Remark 3.6** *The original form of the above states that*

$$k(G, B, d(B), [\pm r], \sigma) = k(N_G(D), b, d(b), [\pm r], \sigma)$$

*for all $r$ and $\sigma \in \mathcal{H}$. However, it can be seen that this is equivalent to Conjecture 3.5. One can see it by using the theory of Burnside rings.*

Of course, we have one more conjecture.

**Conjecture 3.7** (August 2002) *Let $B$ be a block of $G$ with defect group $D$. Suppose that $D \neq \{1\}$ and $O_p(G) = \{1\}$. Then,*

$$\sum_{\underline{C} \in \mathcal{R}/G} (-1)^{|\underline{C}|} k(N_G(\underline{C}), B, d, [\pm r], \sigma) = 0$$

*for all $d \in \mathbb{Z}$, $r \in (\mathbb{Z}/p\mathbb{Z})^*$ and $\sigma \in \mathcal{H}_B$ ?*

-There are also several forms of Conjecture 3.7.

**Remark 3.8** *Suppose that $B$ is principal and $D$ is abelian. Of course, $B$ is $\mathcal{H}$-invariant. Then, if $B$ and its Brauer correspondent $b$ are splendidly equivalent with an $\mathcal{H}$-invariant complex, then Conjecture 3.5 holds for $B$.*

The reason is almost the same as before, since if the complex which gives a splendid equivalence is $\mathcal{H}$-invariant, then the function $\mu$ in Broué's conjecture is $\mathcal{H}$-invariant. Thus $\sigma$-invariant characters must correspond to $\sigma$-invariant characters by the perfect isometry $\varphi$.

This gives rise to the following version of Broué's conjecture.

**Conjecture 3.9** (Galois Invariant Splendid Equivalence Conjecture) *Suppose that a block $B$ of $G$ with defect group $D$ corresponds to a block $b$ of $N_G(D)$ via Brauer's first main theorem and that $D$ is abelian. Does there exist an $\mathcal{H}_B$-invariant bounded complex*

$$\mathbf{C} : \cdots \to C_{i+1} \to C_i \to \cdots$$

*of $R_{\mathcal{P}}G$-$R_{\mathcal{P}}N_G(D)$-bimodules such that each $C_i$ is a $\Delta(D)$-projective $p$-permutation module, where $\Delta(D) = \{(g, g^{-1}) \mid g \in D\}$ ?*

If such a $\mathbf{C}$ exists, then we say that a splendid equivalence between $B$ and $b$ is Galois invariant.

# 4 Cyclic defect case

If a defect group of a block is cyclic, then, we have the following.

**Theorem 4.1** *Suppose that a block $B$ of $G$ has a cyclic defect group. Then, all conjectures appearing in the previous section hold for $B$.*

It suffices to show that Conjectures 3.7 and 3.9 are true. In this situation, Rouquier proved that the splendid equivalence conjecture is true. ([15]) The complex he constructed is $\mathcal{H}_B$-invariant. Thus Conjecture 3.9 is true. For Conjecture 3.7 we use the standard reduction through the normalizer of the unique subgroup of $D$ of order $p$. (§9 of [5])

# 5 A reduction theorem

In [6] Theorem 16.4, Dade proved a reduction theorem. This can be generalized easily to the situation involving $r$. Instead of giving a general result, we consider the following situation.

**5.1** *Let $P$ be a cyclic normal $p$-subgroup of $G$ with $|P| = p^s$. Assume that a Sylow $p$-subgroup $S$ of $G$ is $P \times Q$ for a cyclic subgroup $Q$ of $G$.*

**Proposition 5.2** (i) *Assume 5.1 and suppose that $O_p(G) = P$. (Note that then $P$ is radical.) Then, we have the following.*

$$\sum_{\underline{C} \in \mathcal{R}G/G} (-1)^{|\underline{C}|} k(N_G(\underline{C}), B, d, [\pm r]) = 0$$

*for all $p$-blocks $B$ of $G$ with $d(B) > s$, and for all $d$ and $r$.*

(ii) *Assume 5.1. For any block $B$ of $G$ with defect group $S$, there exists a Galois invariant splendid equivalence between $B$ and its Brauer correspondent.*

For (i), an argument similar to that found in the proof of Theorem 16.4 in [6] gives the result. It can be used when obtaining cancellation results for Dade's conjecture. (Conjecture 3.3.) For (ii) the proof uses Rouquier's construction [15] of complex for cyclic defect case and the argument of Marcus [8] for the existence of extensions of complexes. As an application of (ii) above, we have the following.

**Corollary 5.3** *Assume that a Sylow $p$-subgroup $S$ of $G$ is an elementary abelian of order $p^2$. Then, for a block $B$ with defect group $S$, Conjecture 3.9 implies Conjecture 3.7.*

For the proof of the above, consider a radical $p$-chain starting with $1 < P$ for some $P$ with $|P| = p$. If such a chain exists, then $N_G(P)$ satisfies 5.1 and thus

$$k(N_G(1 < P), B, d, [\pm r], \sigma) = k(N_G(1 < P < S), B, d, [\pm r], \sigma)$$

for all $d$, $r$ and $\sigma \in \mathcal{H}_B$ by Proposition 5.2 (ii). Now, the remaining radical $p$-chains are the trivial one and $1 < S$. Thus, the result holds.

# 6 Examples

Let us give examples of blocks whose defect groups are not abelian. Let $G$ be the sporadic simple Conway's group $Co_2$ or $Co_3$, and let $p = 5$. We verify Conjecture 3.7 in this case. A Sylow 5-subgroup $S$ of $G$ is an extra special group of order $5^3$ and exponent 5. It follows from the Atlas [4] that groups of order $5^2$ are not radical 5-subgroups of $G$. Moreover, among subgroups of order 5, one generated by a $5B$-element is a radical 5-subgroup, while the center of $S$, which is generated by a $5A$-element is not a radical 5-subgroup of $G$. Let $P$ denote a subgroup generated by a $5B$-element and $S'$ a Sylow 5-subgroup of $N_G(P)$. Note that $S'$ is an elementary abelian of order $5^2$. Then the following give representatives of radical 5-chains of $G$.

$$1,\ 1 < P,\ 1 < P < S',\ 1 < S$$

Now, by the argument in the paragraph following Corollary 5.3, in order to verify Conjecture 3.7, it suffices to consider only the trivial chain and $1 < S$. The character tables of $G$ and $N_G(S)$ are found in [4] and [12], respectively. The principal 5-block $B$ of $G$ is the only 5-block of $G$ with defect group $S$ and $N_G(S)$ has of course only the principal block. The numbers of relevant characters are shown as follows.

| $(d, \pm r)$ | $(3,\pm1)$ | $(3,\pm2)$ | $(2,\pm1)$ | $(2,\pm2)$ |
|---|---|---|---|---|
| $k(Co_2, B, d, [\pm r])$ | 10 | 10 | 3 | 4 |
| $k(N_{Co_2}(S), B, d, [\pm r])$ | 10 | 10 | 3 | 4 |
| $k(Co_3, B, d, [\pm r])$ | 10 | 10 | 2 | 4 |
| $k(N_{Co_3}(S), B, d, [\pm r])$ | 10 | 10 | 2 | 4 |

For $Co_2$, irreducible characters in $B$ with defect 3 are all $\mathcal{H}$-invariant. Let us consider those with defect 2. In the notation of the Atlas [4], we have

$$\begin{aligned}\mathrm{Irr}(Co_2, B, 2, [\pm1]) &= \{\chi_{12}, \chi_{13}, \chi_{28}\},\\ \mathrm{Irr}(Co_2, B, 2, [\pm2]) &= \{\chi_{31}, \chi_{32}, \chi_{45}, \chi_{55}\},\end{aligned}$$

with $\chi_{12}(1) = \chi_{13}(1) = 10395 = 2079\cdot5$, $\chi_{28}(1) = 212520 = 42504\cdot5$, $\chi_{31}(1) = \chi_{32}(1) = 239085 = 47817\cdot5$, $\chi_{45}(1) = 637560 = 127512\cdot5$, $\chi_{55}(1) = 1943040 = 388608\cdot5$. Among those, only $\chi_{12}$, $\chi_{13}$, $\chi_{31}$, $\chi_{32}$ have irrational values involving $\sqrt{-15}$. It follows that, if $\sigma \in \mathcal{H}$ sends $\sqrt{-15}$ to $-\sqrt{-15}$, then $\chi_{12}^\sigma = \chi_{13}$ and $\chi_{31}^\sigma = \chi_{32}$.

On the other hand, characters of $N_{Co_2}(S)$ with defect 3 are all $\mathcal{H}$-invariant, and in the notation of [12], we have

$$\begin{aligned}\mathrm{Irr}(N_{Co_2}(S), B, 2, [\pm1]) &= \{\chi_{17}, \chi_{18}, \chi_{19}\},\\ \mathrm{Irr}(N_{Co_2}(S), B, 2, [\pm2]) &= \{\chi_{24}, \chi_{25}, \chi_{26}, \chi_{27}\},\end{aligned}$$

with $\chi_{17}(1) = \chi_{18}(1) = \chi_{19}(1) = 20 = 4\cdot5$, $\chi_{24}(1) = \chi_{25}(1) = \chi_{26}(1) = 40 = 8\cdot5$, $\chi_{27}(1) = 60 = 12\cdot5$. Among those, only $\chi_{17}$, $\chi_{18}$, $\chi_{24}$, $\chi_{25}$ have irrational values involving $\sqrt{-15}$, and if $\sigma \in \mathcal{H}$ sends $\sqrt{-15}$ to $-\sqrt{-15}$, then $\chi_{17}^\sigma = \chi_{18}$ and $\chi_{24}^\sigma = \chi_{25}$. Hence, we have

$$k(Co_2, B, d, [\pm r], \sigma) = k(N_{Co_2}(S), B, d, [\pm r], \sigma)$$

for all $d$, $r$ and $\sigma$.

For $Co_3$, in the notation of [4] we have the following.

$$\begin{aligned}\mathrm{Irr}(Co_3, B, 3, [\pm1]) &= \{\chi_2, \chi_3, \chi_4, \chi_{18}, \chi_{19}, \chi_{21}, \chi_{25}, \chi_{32}, \chi_{36}, \chi_{38}\},\\ \mathrm{Irr}(Co_3, B, 3, [\pm2]) &= \{\chi_1, \chi_6, \chi_7, \chi_8, \chi_9, \chi_{13}, \chi_{14}, \chi_{33}, \chi_{34}, \chi_{42}\},\\ \mathrm{Irr}(Co_3, B, 2, [\pm1]) &= \{\chi_{27}, \chi_{30}\},\\ \mathrm{Irr}(Co_3, B, 2, [\pm2]) &= \{\chi_{10}, \chi_{11}, \chi_{15}, \chi_{40}\},\end{aligned}$$

with $\chi_{10}(1) = \chi_{11}(1) = 3520 = 704\cdot5$, $\chi_{15}(1) = 8855 = 1771\cdot5$, $\chi_{27}(1) = 57960 = 11592\cdot5$, $\chi_{30}(1) = 80960 = 16192\cdot5$, $\chi_{40}(1) = 249480 = 49896\cdot5$. Among those only $\chi_6$, $\chi_7$, $\chi_{18}$, $\chi_{19}$ have irrational values involving $\sqrt{-11}$, and $\chi_{10}$ and $\chi_{11}$ have those involving $\sqrt{-5}$. Any $\sigma \in \mathcal{H}$ leaves $\sqrt{-11}$ invariant, since $\sqrt{-11}$ is expressed as the

Gauss sum and 5 is a square in $\mathbb{Z}/11\mathbb{Z}$. Moreover, it follows that, if $\sigma \in \mathcal{H}$ sends $\sqrt{-5}$ to $-\sqrt{-5}$, then $\chi_{10}^{\sigma} = \chi_{11}$.

On the other hand, for $N_{Co_3}(S)$, in the notation of [12] we have the following.

$$\begin{aligned}
&\mathrm{Irr}(N_{Co_3}, B, 3, [\pm 1]) = \{\chi_9, \chi_{10}, \chi_{11}, \chi_{12}, \chi_{13}, \chi_{14}, \chi_{15}, \chi_{16}, \chi_{17}, \chi_{18}\},\\
&\mathrm{Irr}(N_{Co_3}, B, 3, [\pm 2]) = \{\chi_1, \chi_2, \chi_3, \chi_4, \chi_5, \chi_6, \chi_7, \chi_8, \chi_{23}, \chi_{24}\},\\
&\mathrm{Irr}(N_{Co_3}, B, 2, [\pm 1]) = \{\chi_{25}, \chi_{26}\},\\
&\mathrm{Irr}(N_{Co_3}, B, 2, [\pm 2]) = \{\chi_{19}, \chi_{20}, \chi_{21}, \chi_{22}\}.
\end{aligned}$$

There are 14 characters having irrational values. However, they are $\mathcal{H}$-invariant except for $\chi_{19}$ and $\chi_{20}$, which have values involving $\sqrt{-5}$. It follows that, if $\sigma \in \mathcal{H}$ sends $\sqrt{-5}$ to $-\sqrt{-5}$, then $\chi_{19}^{\sigma} = \chi_{20}$. Hence, we have

$$k(Co_3, B, d, [\pm r], \sigma) = k(N_{Co_3}(S), B, d, [\pm r], \sigma)$$

for all $d$, $r$ and $\sigma$.

Concerning sporadic simple groups, several results are known. In [14] Rouquier verified Perfect Isometry Conjecture for all principal blocks with abelian defect groups. In [7], Isaacs and Navarro confirmed the 2001 version of Alperin-McKay-Isaacs-Navarro Conjecture in the group form using [21]. Up to now, Conjecture 3.3 was verified for all primes for sporadic simple groups except for $J_4$, $Fi_{24}'$, $BM$, $M$. Conjectures 3.7 and 3.9 have not been verified in almost all cases. For current situation of Broué's conjecture, see, for example, 5.2 of [16]. In particular, the results on Broué's conjecture for sporadic simple groups include those for $J_1$ for $p = 2$, $M_{11}$, $M_{22}$, $M_{23}$, $ON$, $HS$ for $p = 3$ and $J_2$ for $p = 5$.

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